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THE ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF DIFFERENTIAL EQUATIONS CONTAINING LARGE AND RAPIDLY CHANGING COEFFICIENTS

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THE ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF DIFFERENTIAL EQUATIONS CONTAINING LARGE AND RAPIDLY CHANGING COEFFICIENTS

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ABSTRACT. The asymptotic behavior of the solutions of differential equations with large and rapidly varying coefficients is investigated for the case of simple second-order equations. The methods can be generalized directly to cover a wide class of higher order equations.

Various problems in mathematical physics make it necessary for us to deal /247* with equations whose coefficients (including those contained in the right side) or their derivatives become large. These include, for example, problems with "potential wells" or "barriers" in quantum mechanics, problems on the propagation of electromagnetic waves through the boundary of a region with high conductivity (M. A. Leontovich conditions (ref. 1), problems with infinitely large right sides in infinitely narrow bands of the simple-or double-layer type. Such problems, in the first approximation, are reduced to certain boundary value problems with boundary conditions or with "junction" conditions on surfaces where the coefficients increase rapidly or are discontinuous. In such problems the investigation of the asymptotic nature of the solution near the specified boundary is basically quite localized. Usually, in this case, the variation in solutions in the direction transverse to the boundary takes place more rapidly than in the tangential direction. Therefore, in constructing the asymptotic solution it is important that we correctly isolate the principal "transverse" part of the operator in a simpler form than that of the operator as a whole. Methods of this type have been applied in problems where higher derivatives have small coefficients (refs. 2 and 3) and in problems with oscillating boundary conditions (refs. 4 and 5). For the sake of brevity and clarity we shall illustrate the asymptotic methods by using simple examples of second order equations; these can be reduced directly to a wide class of higher-order equations.

2. Problems with large coefficients in a constant subdomain.

Problem 1. Let us assume that Γ is a smooth plane curve which bounds the domain Q. Let us consider the equation

$$L_\epsilon u_\epsilon \equiv \Delta u_\epsilon - k_\epsilon(x, y) u_\epsilon = h(x, y), \tag{1}$$

where $k_\epsilon(x,y)=c(x,y)\geq 0$ inside Q and $k_\epsilon(x,y)=c_1(x,y)|\epsilon^2$, $c_1\geq\alpha^2>0$ outside Q; $h(x,y)\equiv 0$ outside Q. We seek the asymptotic expansion of the solutions of equation (1) under conditions that u_ϵ and $\partial u_\epsilon/\partial n$ are continuous on Γ and that u_ϵ

* Numbers in the margin indicate pagination in original foreign text.

becomes equal to zero at infinity. We introduce the following coordinates near Γ: ρ- the distance (with a sign) along the normal to Γ, φ- the coordinate of the base of the normal on Γ. By expressing the operator L_ϵ in terms of these coordinates, by changing the variable such that $t=\rho/\epsilon$ (see (3)), and by expanding the coefficients of the equation in powers of $\rho=t\epsilon$, we obtain, outside Q, (when $\rho>0$)

$$L_\epsilon u_\epsilon \equiv \frac{1}{\epsilon^2}[L_0 u_\epsilon + \epsilon L_1 u + \ldots] = 0, \tag{2}$$

where

$$L_0 u \equiv \frac{\partial^2 u}{\partial t^2} - \beta^2(\varphi) u. \tag{3}$$

As the first approximation of the solution for equation (2) we take the solution u_0 of the boundary layer type equation $L_0 u_0=0$; it has the form /248

$$u_0 = C(\varphi) e^{-\beta t} = C(\varphi) e^{-\beta\rho/\epsilon}, \quad \frac{\partial u_0}{\partial \rho} = -\frac{\beta}{\epsilon} u_0. \tag{4}$$

We anticipate that $\partial u_\epsilon/\partial\rho|_{\rho=-0}$ $(= \partial u_\epsilon/\partial\rho|_{\rho=+0})$ is bounded when $\epsilon\to 0$. Then it follows from (4) that for the first term of the asymptotic expansion

As a result of this it can be shown that the first approximation u_0 inside Q is equal to the solution of the Dirichlet's problem for (1) under condition $u_0|_\Gamma=0$; outside Q $u_0\equiv 0$. The next term u_1 of the asymptotic expansions $u_\epsilon=u_0+\epsilon u_1+\ldots$ is obtained from the conditions that $u_0+\epsilon u_1$ is continuous on Γ and that $\frac{\partial}{\partial\rho}(u_0+\epsilon u_1)$ is continuous with an accuracy determined by quantities of order ϵ. To find u_1 we solve equation $L_0 u_1 = L_1 u_0 \equiv 0$ outside Q with the boundary condition $\partial u_1/\partial t|_{t=+0} = \partial u_0/\partial_\rho|_{\rho=0}$. From the established u_1 outside Q we find u_1 inside Q as the solution of equation (1) in Q with $h\equiv 0$ when $u_1/_{\rho=-0}=u_1/_{\rho=+0}$. Thus $u_0+\epsilon u_1$ is continuous but has a discontinuity of order ϵ in its derivative with respect to ρ on Γ. By continuing this process we can obtain an asymptotic approximation of the n-th order ($\tilde{u}_n = u_0 + \epsilon u_1 + \ldots + \epsilon^n u_n$). In this case $\tilde{u}_n$ is continuous on Γ, has a discontinuity in $\partial u_n/\partial\rho$ of order ϵ^n when $\rho=0$ and satisfies (1) with an accuracy determined by quantities of order ϵ^{n+1} in Q, ϵ^{n-1} outside Q. The residual term of the asymptotic expansion is evaluated by using the method in reference 3 (compare with reference 6).

Problem 2. As a second example, we consider an equation which coincides with (1) inside Q and which has the form

$$L_\varepsilon u_\varepsilon \equiv \varepsilon^\alpha \Delta u_\varepsilon - A^2 \varepsilon^{-\beta} u_\varepsilon = 0 \text{ outside } Q \quad (A \geqslant a^2 > 0). \tag{5}$$

The junction conditions on Γ are as follows: u_ε is continuous on Γ and the "fluxes" are equal

$$\left.\frac{\partial u}{\partial \rho}\right|_{\rho=-0} = \varepsilon^\alpha \left.\frac{\partial u}{\partial \rho}\right|_{\rho=+0}. \tag{6}$$

By repeating the preceding steps we find that the following boundary conditions are satisfied for the first approximation u_0 which, inside Q, is the solution of equation (1): 1) when $\alpha<\beta$ the condition of the first boundary value problem is $u_0|_{\rho=-0} = 0$, 2) when $\alpha \geq \beta$ the condition of the second boundary value problem is $\partial u_0/\partial\rho|_{\rho=-0} = 0$, 3) when $\alpha=\beta$ the condition of the third boundary value problem is $\partial u_0/\partial\rho|_{\rho=-0} + A u_0|_{\rho=-0} = 0$. In the case of (1), u_0, outside Q, is equal to zero while in cases 2) and 3) it is a function of the boundary layer type. The subsequent approximations are constructed as above.

These same methods are applicable in the case when the coefficients are complex (as is the case in the theory of the propagation of electromagnetic waves (ref. 1)) and when the domain Q is not bounded. We also note that they are also applicable in the case when A, for example, is a positively determined differential operator with respect to $\partial/\partial\varphi$.

3. Equations with rapidly varying coefficients. Let us assume that the coefficients of the operator L, everywhere outside the narrow band near Γ, are finite together with their derivatives and in the ε- neighborhood of Γ they have large derivatives and can become large themselves. The following pertain to this situation: problems for equations with discontinuous coefficients*, which are considered as the limiting cases of problems with rapidly varying coefficients; problems in which the coefficients and the right sides become infinitely /249 large in the infinitely small neighborhood of Γ; and various combined problems.

Problem 3. Let us assume that Q_1 is a domain bounded by the curve Γ_1 which contains the line Γ; L is an operator which is assigned in Q_1, with coefficients which, generally speaking, are discontinuous on Γ and smooth in Q-Γ, where L may be written, for example, in the "divergent" form

*It follows from the presentation made below that the formulation of boundary conditions on Γ for such an equation depends on what limiting transition is used to obtain this equation. Roughly speaking different problems differ from each other by the presence of certain δ-type functions or their derivatives in the coefficients, which are concentrated on Γ.

$$Lu \equiv \frac{\partial}{\partial\rho}\left[a(\rho,\varphi)\frac{\partial}{\partial\rho}(b(\rho,\varphi)u)\right] + 2\frac{\partial}{\partial\rho}\left(c\frac{\partial u}{\partial\varphi}\right) + \ldots + fu = h. \tag{7}$$

Let us consider the functions $a_\varepsilon(\rho,\varphi)$, $b_\varepsilon(\rho,\varphi)$, $\ldots$, which are smooth in Q, which coincide with $a(\rho,\varphi)$, $b(\rho,\varphi)$, $\ldots$ when $|\rho|\geq\varepsilon$, and the operator

$$L_\varepsilon u_\varepsilon \equiv \frac{\partial}{\partial\rho}\left[a_\varepsilon(\rho,\varphi)\frac{\partial}{\partial\rho}(b_\varepsilon(\rho,\varphi)u_\varepsilon)\right] + \ldots = h_\varepsilon. \tag{8}$$

In this case $a_\varepsilon(\rho,\varphi) = a_0(\rho/\varepsilon,\varphi) + \rho a_1(\rho/\varepsilon,\varphi) + \ldots + \rho^n a_n(\rho/\varepsilon,\varphi) + \rho^{n+1}a^{n+1}(\varepsilon,\rho,\varphi)$, where the functions $\alpha_0(t,\varphi)$, $\alpha_1(t,\varphi)$ are selected in such a way that when $t=\pm 1$ ($\rho=\pm\varepsilon$) the functions $\alpha_\varepsilon(\rho,\varphi)$ and the n derivatives with respect to ρ coincide with $\alpha(\pm,\varepsilon,\varphi)$ and with its respective derivatives with respect to ρ. In particular $a_0(-1,\varphi) = a(-0,\varphi)$, $a_0(+1,\varphi) = a(+0,\varphi)$. The other coefficients $b_\varepsilon(\rho,\varphi)$, ... are determined in the same manner. For the sake of simplicity we shall limit ourselves to the case when expressions (7) and (8) contain only the first terms, when the coeffici$u_0|_{\rho=+0}=0$, $C(\varphi)=$ $\backslash$d on φ and when $h\equiv h_\varepsilon\equiv 0$ (the general case is investigated in the same manner). Changing to the variable $t=\rho|\varepsilon$ at the pole $|\rho|\leq\varepsilon$, we obtain

$$L_\varepsilon u \equiv \varepsilon^{-2}\{L_0 u + \varepsilon L_1 u + \ldots\}, \quad \text{where} \quad L_0 u \equiv \frac{d}{dt}\left[a_0(t)\frac{d}{dt}(b_0(t)u)\right].$$

At the pole $|\rho|\leq\varepsilon$ equation (8) is replaced approximately by $\tilde{L}_0 u_0=0$. From this we obtain the first approximation if we note that when $|\rho|=\varepsilon\tilde{u}_0$ must be joined to the solution u of equation (7) which is bounded together with its derivatives this means that $d\tilde{u}_0/dt|_{t=\pm 1}=O(\varepsilon)$). We have

$$\begin{gathered} b_0(-1)\tilde{u}_0(-1) = b_0(+1)\tilde{u}_0(+1), \\ a_0(-1)b_0(-1)\frac{\partial\tilde{u}_0(-1)}{\partial t} = a_0(+1)b_0(+1)\frac{\partial\tilde{u}_0(+1)}{\partial t}, \end{gathered} \tag{9}$$

where it is assumed that $b_t'(+1)=b_1'(-1)=0$. These considerations lead to the following proposition that can be proved: the first term u_0 of the asymptotic solution u_ε of equation (8), for some assigned condition on Γ_1 is obtained in the following manner: we solve equation (7) Lu=0 under conditions on Γ_1 and for the junction conditions on Γ: $b(+0)u(+0) = b(-0)u(-0)$, $a(+0)b(+0)u'_\rho|_{\rho=+0} = a(-0)b(-0)u'_\rho|_{\rho=-0}$ and assume that $u_0\equiv u$ outside the band $|\rho|\leq\varepsilon$. The function u_0 satisfies conditions (9) at the points $\rho=\pm\epsilon$ with an accuracy determined by ϵ. In the band $|\rho|<\epsilon$ u_0 is determined as the solution of the ordinary equation $L_0u_0=0$, which is selected in such a way that u_0 and its derivative are continuous when $\rho=-\epsilon$. Then, as calculation steps show, we obtain a discontinuity of order ϵ in u_0 and $u'_{0\rho}$ when $\rho=+\epsilon$. The next approximation is constructed in the same general manner. After n steps we obtain an approximate solution which has closure errors of the order ϵ^{n+1} when $|\rho|>\epsilon$ and of order ϵ^{n-1} when $|\rho|<\epsilon$ and discontinuities in the function and its derivative of order ϵ^{n+1} when $|\rho|=\epsilon$.

Problem 4. Let us illustrate the construction of the asymptotic expansion for an infinitely narrow barrier. Let us assume that everywhere in Q_1

$$Lu \equiv \Delta u - a(\rho, \varphi)\, u = h. \tag{10}$$

Let $L_\epsilon u \equiv Lu$ when $|\rho|\geq\epsilon$ and let /250

$$L_\varepsilon u_\varepsilon \equiv \left(\frac{\partial}{\partial\rho} + \alpha(\varepsilon)\, b\left(\frac{\rho}{\varepsilon}, \varphi\right)\right) L_1 u_\varepsilon + L_2 u_\varepsilon = h_\varepsilon, \tag{11}$$

when $|\rho|<\epsilon$ where L_1 and L_2 are differential operators of the first order. Let $\alpha(\epsilon)=\epsilon^{-1}$. Then the first approximation u_0 of the function $u_\epsilon (h_0=h)$ is obtained when equation (10) is solved under conditions specified on Γ_1 and under the following junction conditions on Γ:

$$u_0(+0, \varphi) = u_0(-0, \varphi), \quad L_1 u_0|_{\rho=-0} = C L_1 u_0|_{\rho=+0}, \; C = \exp \int_{-1}^{+1} b(t, \varphi)\, dt.$$

If, on the other hand $\alpha(\epsilon)=O(1)$, $L_1= \partial/\partial\rho + \ldots$, h_ε has the form of the function which approximates the δ-type function with respect to ρ or the δ'-type function, then in the first approximation to u_ϵ we obtain a function u_0 which has a corresponding discontinuity in the normal derivative or in the function itself. This corresponds to the known properties of classical potentials for simple and double layers. The subsequent approximations are constructed in a similar manner.

4. Let us consider the general problem in which the change of variables $t=\rho|\epsilon$ produces a splitting of form (2) of operator L_ϵ in the band $|\rho| \leqslant \varepsilon$: $L_\varepsilon u_\varepsilon =$ [illegible] $[L_0 \tilde{u}_\varepsilon + \varepsilon L_1 u_\varepsilon + \ldots]$; L_0, L_1 do not contain terms which have derivatives with respect to φ.

We designate by $v_0(t)$, $w_0(t)$ the basic system of solutions of the second-order equation $L_0 u=0$, where $v_0(-1)=1, v_0'(-1)=0$, $w_0(-1), w_0(-1)=0$, $w_0'(-1)=1$. Let us also assume that v_1 is the solution of equation $L_0 v_1=-L_1 v_0, v_1(0)=v_0'(0)=0$. The junction conditions for the first approximation u_0 of the function u_ϵ satisfying equation $L_\epsilon u_\epsilon=0$ will have the form

$$v_0(1)\, u_0|_{-0} = u_0|_{+0}, \quad [\varepsilon^{-1} v_0'(1) + v_1'(1)]\, u_0|_{-0} + w_0'(1)\, u_0'|_{-0} = u_0'|_{+0}.$$

When $v_0'(1) \neq 0$ we shall have the conditions: $u_0|_{-0} = u_0|_{+0} = 0$, $w_0'(1)\, u_0'|_{-0} = u_0'|_{+0}$; when $v_0'(1)=0$ we shall have the conditions: $v_0(1)\, u_0|_{-0} = u_0|_{+0}$, $v_1'(1)\, u_0|_{-0} + w_0'(1)\, u_0'|_{-0} = u_0'|_{+0}$. Similar arguments can be extended to higher order partial differential equations with any number of variables.

5. There is also an intermediate class of problems in which the width of the band where the coefficients vary rapidly also approaches zero but does so more slowly than in the cases considered above. In this case the width of the band may be sufficient to provide for the formation of boundary layer phenomenon and associated discontinuities produced in the limit.

6. It is clear from the above how we must proceed with the solution of the inverse problem: when we are required to use assigned boundary conditions or conditions at the junction to construct equations in a wider region with continuous coefficients, which are functions of ϵ, in such a way that the solution of the initial problem plays the role of the first term in the asymptotic solution of the equation with ϵ. (For example, the basic boundary value problems in Q for the elliptic equation (1) may be simulated as shown in problems 1, 2, and furthermore by combining problems 1, 2 and 3 it is possible to obtain an equation with continuous coefficients $\tilde{L}_\epsilon u_\epsilon=h$, which simulates these problems.)

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